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On construction of equitable social welfare orders on infinite utility streams[☆]

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HIGHLIGHTS

- Every social welfare order (SWO) satisfying Hammond Equity and Strong Pareto is non-constructive.
- For domain set $Y = [0, 1]$, every SWO satisfying Pigou–Dalton transfer principle is non-constructive.
- Provides an example of a SWO which can be represented, but which cannot be constructed.

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ABSTRACT

This paper studies the nature of social welfare orders on infinite utility streams, satisfying the consequentialist equity principles known as Hammond Equity and the Pigou–Dalton transfer principle. The first result shows that every social welfare order satisfying Hammond Equity and the Strong Pareto axioms is non-constructive in nature for all non-trivial domains, Y. The second result shows that, when the domain set is $Y = [0, 1]$, every social welfare order satisfying the Pigou–Dalton transfer principle is non-constructive in nature. Specifically, in both results, we show that the existence of the appropriate social welfare order entails the existence of a non-Ramsey set, a non-constructive object. The second result also provides an example of a social welfare order which can be represented, but which cannot be constructed.

1. Introduction

The subject matter of "intergenerational equity", which has received considerable attention in economics and philosophy, is concerned with the important question of how to treat the well-being of future generations relative to the well-being of those living at present.[1]

In his discussion of the concept of intergenerational equity, Ramsey (1928) maintained that discounting one generation's utility relative to another's is "ethically indefensible", and something that "arises merely from the weakness of the imagination". In the literature on intertemporal social choice, Diamond (1965) formalized the concept of "equal treatment" of all generations (present and future) in the form of an *Anonymity Axiom* on social preferences. This requires that society should be indifferent between two streams of well-being, if one is obtained from the other by interchanging the levels of well-being of any two generations.

In the context of a society where the concern for generations extends over an infinite future, we are led to the question of evaluating infinite utility streams consistently with social preferences which respect the Anonymity axiom. There is, of course, no difficulty in doing this, since the social preference relation which evaluates all infinite utility streams as indifferent satisfies the Anonymity axiom trivially. Such social preference relations, however, are of no interest in social decision making. Clearly, one would *also* like the social preference relation to exhibit *some sensitivity* to individual utility levels in the infinite utility streams. This sensitivity is usually captured in some form of the Pareto principle: society should consider one stream of well-being to be superior to another if at least one generation is better off and no generation is worse off in the former compared to the latter. The various forms of this principle that have been proposed in the context of infinite

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1 See the recent comprehensive survey on intergenerational equity by Asheim (2010).

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utility streams include the Strong Pareto axiom, the Infinite Pareto axiom and the Weak Pareto axiom.[2] However, as soon as one adds such sensitivity requirements to the Anonymity axiom, fundamental difficulties arise in the consistent evaluation of infinite utility streams.

This issue has received considerable attention in the literature on intertemporal welfare economics, and we provide a brief overview. For this purpose, we use the framework that has become standard in this literature. We consider the problem of defining social welfare orders on the set X of infinite utility streams, where this set takes the form of $X = Y^{\mathbb{N}}$, with Y denoting a non-empty set of real numbers and $\mathbb{N}$ the set of natural numbers.

In a seminal contribution, Diamond (1965) showed that there does not exist any continuous social welfare order satisfying the Anonymity and Strong Pareto axioms (where continuity is defined with respect to the sup metric), when Y is the closed interval $[0, 1]$. A social welfare order satisfying the Strong Pareto axiom and the continuity requirement is representable by a social welfare function which is continuous in the sup metric, when Y is the closed interval $[0, 1]$. Thus, Diamond's result also implies that there does not exist any social welfare function satisfying the Anonymity and Strong Pareto axioms, which is continuous in the sup metric, when Y is the closed interval $[0, 1]$.

Basu and Mitra (2003) showed that this last statement can be refined as follows: there does not exist any social welfare function satisfying the Anonymity and Strong Pareto axioms. Another way of stating this is that there does not exist any *representable* social welfare order satisfying the Anonymity and Strong Pareto axioms. That is, by directly imposing the requirement of representability of the social welfare order, one can dispense with the continuity requirement in Diamond's result. The impossibility result established by Basu and Mitra (2003) continues to hold when the Strong Pareto axiom is weakened to the Infinite Pareto axiom (Crespo et al., 2009), or further weakened to the Weak Pareto axiom (Basu and Mitra, 2007; Dubey and Mitra, 2011).

If one does not require representability of the social welfare order, it is possible to show the compatibility of the Anonymity and Strong Pareto axioms. Svensson (1980) established this important result, assuming Y to be the closed interval $[0, 1]$. However, his possibility result uses the variant of Szpilrajn's Lemma given in Arrow (1951), and so the social welfare order is non-constructive; it cannot be used by policy makers for social decision making. Although this is an observation about the particular social welfare order proposed by Svensson, it led to a conjecture by Fleurbaey and Michel (2003) that there exists no explicit description (that is, avoiding the axiom of choice or similar contrivances) of an ordering which satisfies the Anonymity and Weak Pareto axioms. Subsequently, Zame (2007) and Lauwers (2010) showed (by using different definitions of "non-constructive" devices) that it is not possible to obtain a social welfare order satisfying the Anonymity and Strong Pareto axioms without resorting to some non-constructive[3] device. In fact, Lauwers (2010) established his result in the case where the Strong Pareto axiom is replaced by the Intermediate Pareto (which requires both the Infinite Pareto and Monotonicity axioms) axiom. The result of Lauwers continues to hold when the Intermediate Pareto axiom is replaced by the weaker sensitivity requirement of the Weak Pareto axiom. This was established by Dubey (2011), using a variation of the technique introduced by Lauwers (2010), thereby confirming the Fleurbaey–Michel conjecture.

The brief summary of results presented in the previous two paragraphs pertains to the case where the domain set Y is the closed interval $[0, 1]$. This summary would have to be modified appropriately for other domain sets. Let us note that the impossibility results of Basu and Mitra (2003) and of Crespo et al. (2009) hold for any domain set Y consisting of at least two elements, so there is nothing more to be said in those cases. When the sensitivity requirement is weakened to Weak Pareto, however, there is a richer set of possibilities. Basu and Mitra (2007) showed that there exists a representable social welfare order satisfying the Anonymity and Weak Pareto axioms when Y is a subset of $\mathbb{N}$. Following up on this result, Dubey and Mitra (2011) completely characterized the domain sets Y for which there exists a representable social welfare order satisfying the Anonymity and Weak Pareto axioms. These are precisely those domain sets Y which *do not* contain any set of the *order type* of the set of positive and negative integers. Furthermore, in those cases where the domain set Y does not have this property, so that there is no representable social welfare order satisfying the Anonymity and Weak Pareto axioms, it is also not possible to construct a social welfare order satisfying the Anonymity and Weak Pareto axioms (Dubey, 2011).

It will be noted that the *representability* of a social welfare order is conceptually different from the issue of whether it can be *constructed*. The latter concept involves the restriction that use of the axiom of choice or a similar contrivance is forbidden. In the former concept, there is the restriction that the order must have a real-valued representation, but this representation itself might be obtained by using the axiom of choice or a similar device. Nevertheless, in the literature summarized above, it turns out that whenever representation of a social welfare order is actually possible (namely, cases in which the social welfare order is required to satisfy the Anonymity and Weak Pareto axioms, and the domain set Y does not contain any set of the order type of the set of positive and negative integers), one can explicitly write down the appropriate social welfare function (Dubey and Mitra, 2011). And, when representation of the social welfare order is not possible, then construction of the social welfare order is not possible either (Dubey, 2011). It is convenient to refer to this last observation as a *correspondence principle* for social welfare orders (satisfying the Anonymity and Weak Pareto axioms).

The concept of Anonymity is one of *procedural equity*; that is the change involved does not alter the distribution of utilities in the utility stream. Equity principles involving an alteration in the distribution of utilities are called *consequentialist equity* concepts. The equity axiom of Hammond (1976) is one of the key consequentialist equity concepts, the other being the Pigou–Dalton transfer principle. Hammond Equity has several variations which have been discussed in the literature. Strong Equity (see d'Aspremont and Gevers, 1977, Dubey and Mitra, forthcoming-a) and Hammond Equity for the Future (see Asheim et al., 2007, Banerjee, 2006) are notable variations; others are of minor importance conceptually. Altruistic Equity is a variation of the Pigou–Dalton transfer principle (see Hara et al., 2008 and Sakamoto, 2012).

The issues of representation and construction of social welfare orders on infinite utility streams satisfying some notion of consequentialist equity should, of course, be addressed, and several contributions in the literature have been devoted to this topic. However, let us make two initial observations before reviewing this literature. First, some of the concepts of consequentialist equity (like Hammond Equity) do not involve sensitivity, and so (like in the case of Anonymity), we need to supplement the equity notion with an appropriate sensitivity requirement. In other cases (like Strong Equity and the Pigou–Dalton transfer principle), the

2 These sensitivity requirements are in the form of *efficiency* principles. They are, of course, not the only sensitivity principles worth considering. A very weak sensitivity requirement could simply be that there exist two utility streams (say x and y) such that x is socially preferred to y.

3 The words "constructive" and "non-constructive" appear frequently in the rest of the introduction and in subsequent sections. We define these words explicitly in Section 2, drawing on the appropriate mathematical literature.

consequentialist equity concept itself requires some sensitivity, and then it is less clear whether additional requirements are warranted, and if so which additional requirements are incontrovertible. Second, because there are several notions of consequentialist equity, it is more helpful to have a selective review of the literature (which highlights the important new ideas) rather than an exhaustive catalog of results in all cases.

Strong Equity involves comparisons between two utility streams (x and y) in which all generations except two have the same utility levels in both utility streams. Regarding the two remaining generations (say, i and j), one of the generations (say i) is better off in utility stream x, and the other generation (j) is better off in utility stream y, thereby setting up a conflict. The axiom states that if for both utility streams, it is generation i which is worse off than generation j (this, of course, requires us to make intergenerational comparisons of utilities), then generation i should be allowed (on behalf of the society) to choose between x and y. That is, x is socially preferred to y, since generation i is better off in x than in y. In the case of Hammond Equity, in a similar comparison, one makes the weaker statement that x is socially at least as good as y.

We focus first on the literature relating to Hammond Equity. As this concept does not involve sensitivity, we add a sensitivity requirement in the form of the Strong Pareto axiom.[4] Bossert et al. (2007) have shown that when $Y = \mathbb{R}$, there exist social welfare orders on infinite utility streams which satisfy Hammond Equity and the Strong Pareto axiom. On the other hand, Alcantud and Garcia-Sanz (2013) show that any social welfare order satisfying Hammond Equity and Strong Pareto cannot be represented by a real-valued function, if the domain set (Y) consists of at least four distinct elements. That is, an impossibility result arises as soon as we admit a situation in which Hammond Equity can play a role in ranking two utility streams. The existence result of Bossert et al. (2007) uses the variant of Szpilrajn's Lemma given in Arrow (1951), a non-constructive device. This, of course, leaves open the question of whether such social welfare orders can be constructed. We show (in Proposition 1) that in fact the existence of such social welfare orders implies the existence of a non-Ramsey set, which is a non-constructive object. Thus, a *correspondence principle* holds for social welfare orders satisfying Hammond Equity and the Strong Pareto axiom. The situations in which representation of such social welfare orders is not possible (which is *all* non-trivial situations in this case), the construction of such orders is not possible either.[5]

[4] For results on the representation of social welfare orders satisfying Hammond Equity, when the sensitivity requirement of Strong Pareto is replaced by Weak Pareto see Alcantud (2012), Alcantud and Garcia-Sanz (2013) and Dubey and Mitra (forthcoming-b).

[5] Given the strongly negative results noted in the above paragraph, it is of interest to examine whether the situation changes if the equity requirement is strengthened to Strong Equity, while the efficiency principle is taken to be in its weakest form, known as Monotonicity. (This efficiency concept is incontrovertible as it only requires that if no one is worse off (in utility stream x compared to y), then the society as a whole should not be worse off (in utility stream x compared to y)). Actually, the theory (as developed by Dubey and Mitra (forthcoming-a)) turns out to be not only somewhat more subtle in this case, but also quite complete. There exist social welfare functions satisfying the Strong Equity axiom and Monotonicity if and only if the domain set (Y) has at most five distinct elements. Further, when the domain set (Y) has more than five distinct elements, the existence of any social welfare order satisfying the Strong Equity axiom and Monotonicity implies the existence of a non-Ramsey set, a non-constructive object. It is worth pointing out that when representation is possible (that is, when Y has at most five distinct elements), one can explicitly write down the appropriate social welfare function. Also, in this context, the *correspondence principle* holds: if there is no representable social welfare order satisfying the Strong Equity axiom and Monotonicity, then there is no social welfare order, satisfying the Strong Equity axiom and Monotonicity, which can be constructed.

We turn next to the other important consequential equity concept, the Pigou–Dalton transfer principle. This involves comparisons between two utility streams (x and y) in which all generations except two have the same utility levels in both utility streams. Regarding the two remaining generations (say, i and j), one of the generations (say i) is better off in utility stream x, and the other generation (j) is better off in utility stream y, thereby setting up a conflict. If for both utility streams, generation j is at least as well off as generation i (so that we have $y_j > x_j \geq x_i > y_i$) and x can be obtained from y by transferring utility from generation j to generation i (so that $y_j - x_j = x_i - y_i$), then x is socially preferred to y.

Since the Pigou–Dalton transfer principle itself embodies sensitivity, we examine the representation and construction of social welfare orders satisfying just this principle, without adding other requirements. Using the technique introduced by Basu and Mitra (2007, Proposition 1), it has been established by Sakamoto (2012, Proposition 3) that there is a representable social welfare order satisfying the Pigou–Dalton transfer principle, when the domain set is $Y = [0, 1]$. This existence result uses the Axiom of Choice. We show (in Proposition 2) that (when $Y = [0, 1]$) the existence of such a social welfare order implies the existence of a non-Ramsey set, which is a non-constructive object. This result implies that a social welfare order, satisfying the Pigou–Dalton transfer principle, cannot be constructed over infinite utility streams when the domain set is $Y = [0, 1]$.

Proposition 2 has further significance. It provides an important example of a social welfare order which can be represented, but which cannot be constructed. It is, of course, well-known that there are social welfare orders (like the lexicographic order) which can be constructed but not represented. Thus, the correspondence principle between the representation and construction of social welfare orders, (that we have observed above for other equity concepts) is not universally valid.

2. Preliminaries

2.1. Notation

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real numbers and natural numbers respectively. Let T be an infinite subset of $\mathbb{N}$. We denote by $\Omega(T)$ the collection of all infinite subsets of T, and we denote $\Omega(\mathbb{N})$ by Ω. Thus, for any infinite subset T of $\mathbb{N}$, we have $T \subset \mathbb{N}$, and $T \in \Omega$.

For all $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

2.2. Definitions

2.2.1. Social welfare order

Let Y, a non-empty subset of $\mathbb{R}$, be the set of all possible utilities that any generation can achieve. Then $X \equiv Y^{\mathbb{N}}$ is the set of all possible utility streams. If $\langle x_n \rangle \in X$, then $\langle x_n \rangle = (x_1, x_2, \ldots)$, where, for all $n \in \mathbb{N}$, $x_n \in Y$ represents the amount of utility that the generation of period n earns.

We consider binary relations on X, denoted by $\succsim$, with the symmetric and asymmetric parts denoted by $\sim$ and $\succ$ respectively, defined in the usual way. A *social welfare order* (SWO) is a complete and transitive binary relation.

A *social welfare function* (SWF) is a mapping $W : X \to \mathbb{R}$. Given an SWO $\succsim$ on X, we say that $\succsim$ can be *represented* by a real-valued function if there is a mapping $W : X \to \mathbb{R}$ such that for all $x, y \in X$, we have $x \succsim y$ if and only if $W(x) \geq W(y)$.

2.2.2. Efficiency

The following efficiency axioms on social welfare orders are used in the analysis or the discussion of this paper.

Definition 1 (*Monotonicity (MON)*)**.** For $x, y \in X$, if $x \geq y$, then $x \succsim y$.

Definition 2 (*Strong Pareto (SP)*)**.** For $x, y \in X$, if $x > y$, then $x \succ y$.

Definition 3 (*Infinite Pareto (IP)*)**.** For $x, y \in X$, if $x > y$, and there exists a strictly increasing subsequence of natural numbers $\{j_n : n \in \mathbb{N}\}$ such that $x_{j_n} > y_{j_n}$, then $x \succ y$.

Definition 4 (*Weak Pareto (WP)*)**.** For $x, y \in X$, if $x \gg y$, then $x \succ y$.

2.2.3. Equity

The following equity axioms on social welfare orders are used in the analysis or the discussion of this paper. The procedural equity criterion that we will use is Anonymity (also sometimes known as Finite Anonymity).

Definition 5 (*Anonymity (AN)*)**.** If $x, y \in X$, and if there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, and for every $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$, then $x \sim y$.

The consequentialist equity criteria that we will use are:

Definition 6 (*Strong Equity (SE)*)**.** If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succ y$.

Definition 7 (*Hammond Equity (HE)*)**.** If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succsim y$.

Definition 8 (*Pigou–Dalton Transfer Principle (PD)*)**.** If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j \geq x_i > y_i$, and $y_j + y_i = x_j + x_i$ while $y_k = x_k$ for all $k \in N \setminus \{i, j\}$, then $x \succ y$.

2.2.4. Ramsey and non-Ramsey collections of sets

A collection of sets $\Gamma \subset \Omega$ is called *Ramsey*[6] if there exists $T \in \Omega$ such that either $\Omega(T) \subset \Gamma$ or $\Omega(T) \subset \Omega \diagdown \Gamma$. Next we define collection of sets known as non-Ramsey.

Definition 9 (*Non-Ramsey Sets*)**.** A collection of sets $\Gamma \subset \Omega$ is non-Ramsey[7] if for every $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and its complement $\Omega \diagdown \Gamma$.

2.2.5. Non-constructive statements and objects

We will be concerned with statements of the form, "There exists a social welfare order $\succsim$ on X, satisfying property P". A statement of this form is an assertion regarding the existence of an object O: in our case, the object O is a social welfare order $\succsim$ on X, satisfying property P.

Consider then, in general, the statement, "There exists an object O". Such a statement will be called *non-constructive* if (i) it can be established in every model of ZFC set theory (Zermelo–Fraenkel set theory with the axiom of choice AC), but (ii) there is some model of ZF set theory (Zermelo–Fraenkel set theory without AC) in which it cannot be established. In this case, we will also say that the object O is a *non-constructive object*, and that the object O cannot be *constructed*. For the sake of completeness, let us add that we will say that an object O can be *constructed* if the existence of the object O can be established in every model of ZF set theory.

As an example of the use of this terminology, consider the statement, "There exists a non-Ramsey collection of sets $\Gamma \subset \Omega$". The object O, whose existence is being asserted, is "a non-Ramsey collection of sets $\Gamma \subset \Omega$". The statement can be established under ZFC; in fact, it has been established under ZFC by Erdős and Rado (1952). Mathias (1977) has shown that in Solovay's model $\mathfrak{M}_1$ (which satisfies ZF, but violates AC) every collection of sets $\Gamma \subset \Omega$ is Ramsey, and so there is some model of ZF set theory in which the statement cannot be established. Thus, according to our terminology, the statement "There exists a non-Ramsey collection of sets $\Gamma \subset \Omega$" is non-constructive. Furthermore, we can say that the object "a non-Ramsey collection of sets $\Gamma \subset \Omega$" is a non-constructive object, and this object cannot be constructed.

Let us make two remarks regarding our definition. First, our discussion here belongs in the area of mathematics known as *Constructive Mathematics*. It is clear from the literature that there is no consensus about the definitions of various key concepts in this sub-discipline. For an overview, together with a comprehensive set of references to some of the original contributions in this area, see Beeson (1985). Our definition is one which is commonly used by many mathematicians, currently making contributions in set theory and mathematical logic. As an example of its common usage (reflected in the fact that the definition is implicit rather than explicit), we refer the reader to Howard et al. (2001), both for the choice of the title of the paper, and the first paragraph of its introduction.[8]

Second, our definition (and the illustration of its use) assumes some degree of familiarity with Zermelo–Fraenkel set theory, the Axiom of Choice, and Model theory. For an overview of these concepts, we refer the reader to Zame (2007, Section 4); a comprehensive coverage can be found in Chang and Keisler (1992) and Jech (1978).

3. Results and proofs

In this section, we present the results of the paper, dealing with the non-constructive nature of social welfare orders satisfying some well-known consequentialist equity conditions. The statement of the results, together with accompanying discussions, appear in the first subsection. The second subsection contains the complete proofs.

6 If we consider Ω to be a topological space, endowed with the topology inherited from the standard product topology on $\mathbb{R}^{\mathbb{N}}$, then Galvin and Prikry (1973) showed that if $\Gamma \subset \Omega$ is any Borel set, then it is Ramsey; in particular, if $\Gamma \subset \Omega$ is any open set, then it is Ramsey.

7 The concept of non-Ramsey collection of sets was introduced in the literature on ordering infinite utility streams by Lauwers (2010). Erdős and Rado (1952, Example 1, p. 434) have shown, using Zermelo's well-ordering principle (which is known to be equivalent to the Axiom of Choice), that there is a collection of sets $\Gamma \subset \Omega$, which is non-Ramsey.

8 In their definitive study, Howard and Rubin (1998) note that they are "concerned with sentences in the language of set theory which can be proved using the axiom of choice but which are not theorems of set theory with that axiom omitted". They call such sentences "forms of the axiom of choice", and analyze 383 such forms of the axiom of choice. We call such sentences "non-constructive".

3.1. Non-constructive equitable social welfare orders

3.1.1. Hammond equity and Strong Pareto

We focus first on the consequentialist equity principle known as Hammond equity, and investigate the existence of social welfare orders satisfying Hammond Equity together with the efficiency principle known as Strong Pareto, when Y consists of at least four distinct elements.

The result of Bossert et al. (2007, Theorem 2) establishes the existence of a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying Hammond Equity and Strong Pareto axioms, by invoking Arrow's version of Szpilrajn's Lemma; see Arrow (1951, p. 64). In any model of ZFC, Szpilrajn's Lemma holds by the Axiom of Choice (see Jech, 1973, p. 19), and consequently so does Arrow's version of it. Thus, in any model of ZFC, there exists a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying Hammond Equity and Strong Pareto axioms.

The principal result in this subsection (Proposition 1) is that the statement: "There exists a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying Hammond Equity and Strong Pareto axioms" is non-constructive. That is, in terms of the definitions given in Section 2.2.5, a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying Hammond Equity and Strong Pareto axioms cannot be constructed. We establish this by first proving that when there exists a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying Hammond Equity and Strong Pareto axioms, then $\succsim$ can be used to define a set $\Gamma \subset \Omega$, which is non-Ramsey.

We try to explain informally the content and proof of Proposition 1. To this end, observe that a social welfare order $\succsim$ is concerned with ranking of utility streams in $X = Y^{\mathbb{N}}$, and Ramsey or non-Ramsey collections of sets refer to collections of infinite subsets of $\mathbb{N}$; so we need to link the two. We do this by adopting *explicit rules* which determine the assignments of utilities to the various generations, depending on how the set of all generations is partitioned. This procedure is constructive, as it does not use any form of the Axiom of Choice.

To elaborate on the procedure, given any infinite subset $N \subset \mathbb{N}$, we explicitly define a unique finite subset $\overline{N} \subset \mathbb{N}$ with it. Then, we explicitly define $x(N) \in X$ and $y(N) \in X$, associated with N.[9] (These are also written as $x(N, \overline{N})$ and $y(N, \overline{N})$, but they can be written as $x(N)$ and $y(N)$ respectively, since $\overline{N}$ is uniquely defined by an explicit formula as soon as N is specified). Then, we define a subset $\Gamma \subset \Omega$ by:

$$\Gamma \equiv \{N \in \Omega : y(N) \succ x(N)\}.$$

This Γ is a well-defined set, since the preference order $\succsim$ is complete. In Proposition 1(i), we show that this set Γ is a non-Ramsey set.

We now turn to Proposition 1(ii). If a preference order $\succsim$ satisfying Hammond Equity and Strong Pareto can be constructed, then the set Γ can be constructed too, since it is defined in terms of $\succsim$ by using a constructive procedure (as described above). This is a contradiction, since we know from Proposition 1(i) that Γ is a non-Ramsey set, and from the definition and discussion of Section 2.2.5 that consequently Γ cannot be constructed. Thus, a preference order satisfying Hammond Equity and Strong Pareto cannot be constructed.

Proposition 1. *Let $Y \subset \mathbb{R}$ contain at least four distinct elements.*

(i) *If there exists a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying Hammond Equity and Strong Pareto axioms, then $\succsim$ can be used to define a set $\Gamma \subset \Omega$, which is non-Ramsey.*
(ii) *A social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying Hammond Equity and Strong Pareto axioms cannot be constructed.*

Remark. It is known (see Alcantud and Garcia-Sanz, 2013) that any social welfare order satisfying Hammond Equity and Strong Pareto cannot be represented by a real-valued function, if the domain set (Y) consists of at least four distinct elements. That is, as soon as we admit a situation in which Hammond Equity can play a role in ranking two utility streams, it becomes impossible to represent such a social welfare order. Our result in this subsection shows that if the domain set (Y) consists of at least four distinct elements, there is no social welfare order satisfying Hammond Equity and Strong Pareto, which can be constructed. Thus, a *correspondence principle* holds for social welfare orders satisfying Hammond Equity and the Strong Pareto axiom. The situations in which representation of such social welfare orders is not possible (which is *all* non-trivial situations in this case), the construction of such orders is not possible either.

3.1.2. Pigou–Dalton transfer principle

We turn next to the consequentialist equity principle known as the Pigou–Dalton transfer principle, and investigate the existence of social welfare orders satisfying the Pigou–Dalton transfer principle, when Y is the closed interval $[0, 1]$.

The result of Bossert et al. (2007, Theorem 1) establishes the existence of a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying the Pigou–Dalton transfer principle, by invoking Arrow's version of Szpilrajn's Lemma. As noted in the previous subsection, in any model of ZFC, Szpilrajn's Lemma holds and consequently so does Arrow's version of it. Thus, in any model of ZFC, there exists a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying the Pigou–Dalton transfer principle.

The principal result in this subsection (Proposition 2) is that the statement: "There exists a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying the Pigou–Dalton transfer principle" is non-constructive. That is, in terms of the definitions given in Section 2.2.5, a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying the Pigou–Dalton transfer principle cannot be constructed. This is an especially strong negative result since no efficiency principle is imposed on the social welfare order $\succsim$ on X. That is, unlike the result of the previous section, this negative result cannot be attributed to a *conflict* between equity and efficiency principles imposed on a social welfare order.

Our procedure for establishing Proposition 2 is similar to that used to establish Proposition 1. We first prove that when there exists a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying the Pigou–Dalton transfer principle, then $\succsim$ can be used to define a set $\Gamma \subset \Omega$, which is non-Ramsey. If a preference order $\succsim$ satisfying the Pigou–Dalton transfer principle can be constructed, then the set Γ can be constructed too, since it is defined in terms of $\succsim$ by using a constructive procedure. This is a contradiction, since we know from Proposition 2(i) that Γ is a non-Ramsey set, and from the definition and discussion of Section 2.2.5 that consequently Γ cannot be constructed. Thus, a preference order satisfying the Pigou–Dalton transfer principle cannot be constructed.

Proposition 2. *Let Y be the closed interval* $[0, 1]$.

(i) *If there exists a social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying the Pigou–Dalton transfer principle, then $\succsim$ can be used to define a set $\Gamma \subset \Omega$, which is non-Ramsey.*
(ii) *A social welfare order $\succsim$ on $X = Y^{\mathbb{N}}$ satisfying the Pigou–Dalton transfer principle cannot be constructed.*

Remark. Using the technique introduced by Basu and Mitra (2007), it has been established by Sakamoto (2012) that there is a representable social welfare order satisfying the Pigou–Dalton transfer principle, when the domain set Y is $[0, 1]$. (The proof of this

[9] These explicit definitions appear in displays (1) and (2) of the proof of Proposition 1.

existence result uses the Axiom of Choice.) In view of this, Proposition 2 provides an important example of a social welfare order which cannot be constructed, even though it can be represented. It is, of course, well-known that there are social welfare orders (like the lexicographic order) which can be constructed but not represented. Thus, the *correspondence principle* between the representation and construction of social welfare orders, that can be observed for other equity concepts, is not universally valid.

3.2. Proofs of the two results

Proof of Proposition 1. We establish statement (i) of Proposition 1. Statement (ii) then follows from the argument given in Section 3.1.1, just before the statement of Proposition 1.

Let Y contain four distinct elements. Define $Y \equiv \{a, b, c, d\}$, with $a < b < c < d$. Let $N \equiv \{n_1, n_2, n_3, n_4, \ldots\}$ be an infinite subset of $\mathbb{N}$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Let $\overline{N} = \{1, 2, \ldots, n_4 - 1\}$. For any $T \in \Omega(N)$, $T \equiv \{t_1, t_2, t_3, t_4, \ldots\}$ with $t_k < t_{k+1}$ for all $k \in \mathbb{N}$, we partition the set of natural numbers $\mathbb{N}$ in $U = \{t_1, t_1+1, \ldots, t_2-1, t_3, \ldots, t_4-1, \ldots\}$ and $U^c = \mathbb{N} \setminus U$. Let $\overline{U^cT} = U^c \cap \overline{N}$ and $\overline{UT} = U \cap \overline{N}$. Further, let $\overline{U^cN} = U^c \setminus \overline{N}$, and $\overline{UN} = U \setminus \overline{N}$. We define the utility stream $x(T, \overline{N})$ whose components are,

$$x_t = \begin{cases} c & \text{if } t \in \overline{U^cT}, \quad d \quad \text{if } t \in \overline{UT}, \\ a & \text{if } t \in \overline{U^cN}, \quad b \quad \text{if } t \in \overline{UN}. \end{cases} \tag{1}$$

The utility assigned to generations in $\overline{U^cT}$ and $\overline{UT}$ are c, and d respectively. Also the utility assigned to generations in $\overline{U^cN}$ and $\overline{UN}$ are a and b respectively.

The utility stream $y(T, \overline{N})$ is defined using the subset $T \setminus \{t_1\}$ in place of subset T, in following fashion. The two partitions of the set of natural numbers $\mathbb{N}$ are $\widehat{U} = \{t_2, t_2+1, \ldots, t_3-1, t_4, \ldots, t_5-1, \ldots\}$ and $\widehat{U^c} = \mathbb{N} \setminus \widehat{U}$. Let $\widehat{U^cT} = \widehat{U^c} \cap \overline{N}$ and $\widehat{UT} = \widehat{U} \cap \overline{N}$. Further, let $\widehat{U^cN} = \widehat{U^c} \setminus \overline{N}$, and $\widehat{UN} = \widehat{U} \setminus \overline{N}$. We define the utility stream $y(T, \overline{N})$ whose components are,[10]

$$y_t = \begin{cases} c & \text{if } t \in \widehat{U^cT}, \quad d \quad \text{if } t \in \widehat{UT}, \\ a & \text{if } t \in \widehat{U^cN}, \quad b \quad \text{if } t \in \widehat{UN}. \end{cases} \tag{2}$$

As $\overline{N}$ is unique for any N, $x(S, \overline{N})$ and $y(S, \overline{N})$ are well-defined for any $S \in \Omega(N)$.

We will prove a *stronger* result by replacing the Strong Pareto axiom with the *weaker* Infinite Pareto axiom. Let $\succsim$ be a social welfare order satisfying HE and IP. We claim that the collection of sets $\Gamma \equiv \{N \in \Omega : y(N) \succ x(N)\}$ is non-Ramsey. We need to show that for each $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and $\Omega \diagdown \Gamma$. For this, it is sufficient to show that for each $T \in \Omega$, there exists $S \in \Omega(T)$ such that either $T \in \Gamma$ or $S \in \Gamma$, with the either/or being exclusive. Let $T \equiv \{t_1, t_2, \ldots\}$. In the remaining proof we are concerned with infinite utility sequences $x(T, \overline{T})$, $y(T, \overline{T})$ and $x(S, \overline{T})$, $y(S, \overline{T})$ where $S \in \Omega(T)$. For ease of notation, we omit reference to $\overline{T}$. As the binary relation is complete, one of the following cases must arise: (a) $y(T) \succ x(T)$; (b) $x(T) \succ y(T)$; (c) $x(T) \sim y(T)$. Accordingly, we now separate our analysis into three cases.

(a) Let $x(T) \prec y(T)$; that is, $T \in \Gamma$. We drop t_1 and t_{4n+1}, t_{4n+2} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_2, t_3, t_4, t_7, t_8, t_{11}, \ldots\}$. Hence $S \in \Omega(T)$. Let $T_1 \equiv \{t_1, t_1+1, \ldots, t_2-1\}$ and $T_2 \equiv \{t_{4n+1}, t_{4n+1}+1, \ldots, t_{4n+2}-1 : n \in \mathbb{N}\}$. Note that T_2 contains infinitely many elements. Observe that
   (A) for all $t \in T_1, x_t(T) = d > c = y_t(S)$;
   (B) for all $t \in T_2, x_t(T) = b > a = y_t(S)$ and $x_t(S) = b > a = y_t(T)$; and
   (C) for all the remaining $t \in \mathbb{N}, x_t(T) = y_t(S)$ and $x_t(S) = y_t(T)$.
   Then $y(S) \prec x(T)$ and $y(T) \prec x(S)$ by IP. Since $x(T) \prec y(T)$, we get
   $$y(S) \prec x(T) \prec y(T) \prec x(S) \Rightarrow S \notin \Gamma.$$

(b) Let $y(T) \prec x(T)$, or $T \notin \Gamma$. We drop t_1 and t_{4n}, t_{4n+1} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_2, t_3, t_6, t_7, t_{10}, t_{11}, \ldots\}$. Hence $S \in \Omega(T)$. Let $T_1 \equiv \{t_1, t_1+1, \ldots, t_2-1\}$ and $T_2 \equiv \{t_{4n}, t_{4n}+1, \ldots, t_{4n+1}-1 : n \in \mathbb{N}\}$. Note that T_2 contains infinitely many elements. Observe that
   (A) for all $t \in T_1, x_t(T) = d > c = y_t(S)$; and there are finitely many coordinates in T_1;
   (B) for all $t \in T_2, y_t(T) = b > a = x_t(S)$ and $y_t(S) = b > a = x_t(T)$;
   (C) for all coordinates $p \in T_1$, it is possible to pick a coordinate $q \in T_2$, such that $x_p(T) = d > y_p(S) = c > b = y_q(S) > a = x_q(T)$;
   (D) for all the remaining $t \in \mathbb{N}, x_t(T) = y_t(S)$ and $x_t(S) = y_t(T)$.
   So, $x(S) \prec y(T)$ by IP and $x(T) \prec y(S)$ by HE and IP. Since $y(T) \prec x(T)$,
   $$x(S) \prec y(T) \prec x(T) \prec y(S) \Rightarrow S \in \Gamma.$$

(c) Let $x(T) \sim y(T)$ or $T \notin \Gamma$. We drop t_1, t_2, t_3 and t_{4n+2}, t_{4n+3} for all $n \in \mathbb{N}$ from T to obtain $S = \{t_4, t_5, t_8, t_9, \ldots\}$. Hence $S \in \Omega(T)$. Let $T_1 \equiv \{t_1, \ldots, t_2-1; t_3, \ldots, t_4-1\}$, $T_2 \equiv \{t_2, \ldots, t_3-1\}$ and $T_3 \equiv \{t_{4n+2}, \ldots, t_{4n+3}-1 : n \in \mathbb{N}\}$. Note that T_3 contains infinitely many elements. Observe that
   (A) for all $t \in T_2, y_t(T) = d > c = x_t(S)$;
   (B) for all $t \in T_3, y_t(S) = b > a = x_t(T)$ and $y_t(T) = b > a = x_t(S)$;
   (C) for all $t \in T_1, x_t(T) = d > c = y_t(S)$; and there are finitely many coordinates in T_1;
   (D) for all coordinates $p \in T_1$, it is possible to pick a coordinate $q \in T_3$, such that $x_p(T) = d > y_p(S) = c > b = y_q(S) > a = x_q(T)$.
   (E) for all the remaining $t \in \mathbb{N}, x_t(T) = y_t(S)$ and $x_t(S) = y_t(T)$.
   So, $x(S) \prec y(T)$ by IP and $x(T) \prec y(S)$ by HE and IP. Since $x(T) \sim y(T)$, we get
   $$x(S) \prec y(T) \sim x(T) \prec y(S) \Rightarrow S \in \Gamma. \quad \square$$

Proof of Proposition 2. We establish statement (i) of Proposition 2. Statement (ii) then follows from the argument given in sub-Section 3.1.2, just before the statement of Proposition 2.

Let Y contain at least eight distinct elements. Define $Y \equiv \{a, b, c, d, e, f, g, h\}$, with $a < b < c < d < e < f < g < h$, $a+h = b+g$, $c+f = d+e$, $a+d = b+c$, $e+h = f+g$.[11] Let $N \equiv \{n_1, n_2, n_3, n_4, \ldots\}$ be an infinite subset of $\mathbb{N}$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$. Let $\overline{N} = \{1, 2, \ldots, 2(n_4-1)\}$. For any $T \in \Omega(N)$, $T \equiv \{t_1, t_2, t_3, t_4, \ldots\}$ with $t_k < t_{k+1}$ for all $k \in \mathbb{N}$, we partition the set of natural numbers $\mathbb{N}$ in $U = \{2t_1-1, 2t_1, \ldots, 2(t_2-1), 2t_3-1, \ldots, 2(t_4-1), \ldots\}$ and $U^c = \mathbb{N} \setminus U$. Let $\overline{U^cTE} = \{t \in U^c \cap \overline{N} : t \text{ is even}\}$ and $\overline{U^cTO} = U^c \cap \overline{N} \setminus \overline{U^cTE}$. Also, $\overline{UTE} = \{t \in U \cap \overline{N} : t \text{ is even}\}$, $\overline{UTO} = U \cap \overline{N} \setminus \overline{UTE}$, $\overline{U^cN} = U^c \setminus \overline{N}$, and $\overline{UN} = U \setminus \overline{N}$. Further, $\overline{U^cNE} = \{t \in \overline{U^cN} : t \text{ is even}\}$, $\overline{U^cNO} =$

[10] If $n_1 = 1$, then $\{1, \ldots, n_1 - 1\} = \emptyset$. For illustration, for $N = \{1, 2, 3, 4, \ldots\}$, $\overline{N} = \{1, 2, 3\}$ and the two utility streams are $x(N, \overline{N}) = \{d, c, d, a, b, a, \ldots\}$ and $y(N, \overline{N}) = \{c, d, c, b, a, \ldots\}$.

[11] Take $a = 0, b = \frac{1}{8}, c = \frac{2}{8}, d = \frac{3}{8}, e = \frac{4}{8}, f = \frac{5}{8}, g = \frac{6}{8}$ and $h = \frac{7}{8}$ for instance.

$\overline{U^cN} \setminus \overline{U^cNE}$, $\overline{UNE} = \{t \in \overline{UN} : t \text{ is even}\}$, and $\overline{UNO} = \overline{UN} \setminus \overline{UNE}$. We define the utility stream $x(T, \overline{N})$ whose components are,

$$x_t = \begin{cases} c & \text{if } t \in \overline{U^cTO}, \quad f \quad \text{if } t \in \overline{U^cTE}, \\ d & \text{if } t \in \overline{UTO}, \quad e \quad \text{if } t \in \overline{UTE}, \\ a & \text{if } t \in \overline{U^cNO}, \quad h \quad \text{if } t \in \overline{U^cNE}, \\ b & \text{if } t \in \overline{UNO}, \quad g \quad \text{if } t \in \overline{UNE}. \end{cases} \tag{3}$$

The utility assigned to odd and even generations in $U^c \cap \overline{N}$ and $U \cap \overline{N}$ are c, f, d and e respectively. Similarly the utility assigned to odd and even generations in $\overline{U^cN}$ and $\overline{UN}$ are a, h, b and g respectively.

The utility stream $y(T, \overline{N})$ is defined using the subset $T \setminus \{t_1\}$ in place of subset T, in the following fashion. The two partitions of the set of natural numbers $\mathbb{N}$ are $\widehat{U} = \{2t_2 - 1, 2t_2, \ldots, 2(t_3 - 1), 2t_4 - 1, \ldots, 2(t_5 - 1), \ldots\}$ and $\widehat{U^c} == \mathbb{N} \setminus \widehat{U}$. Let $\widehat{U^cTE} = \{t \in \widehat{U^c} \cap \overline{N} : t \text{ is even}\}$ and $\widehat{U^cTO} = \widehat{U^c} \cap \overline{N} \setminus \widehat{U^cTE}$. Also, $\widehat{UTE} = \{t \in \widehat{U} \cap \overline{N} : t \text{ is even}\}$, $\widehat{UTO} = \widehat{U} \cap \overline{N} \setminus \widehat{UNE}$, $\widehat{U^cN} = \widehat{U^c} \setminus \overline{N}$, and $\widehat{UN} = \widehat{U} \setminus \overline{N}$. Further, $\widehat{U^cNE} = \{t \in \widehat{U^cN} : t \text{ is even}\}$, $\widehat{U^cNO} = \widehat{U^cN} \setminus \widehat{U^cNE}$, $\widehat{UNE} = \{t \in \widehat{UN} : t \text{ is even}\}$, $\widehat{UNO} = \widehat{UN} \setminus \widehat{UNE}$. We define the utility stream $y(T, \overline{N})$ whose components are,[12]

$$y_t = \begin{cases} c & \text{if } t \in \widehat{U^cTO}, \quad f \quad \text{if } t \in \widehat{U^cTE}, \\ d & \text{if } t \in \widehat{UTO}, \quad e \quad \text{if } t \in \widehat{UTE}, \\ a & \text{if } t \in \widehat{U^cNO}, \quad h \quad \text{if } t \in \widehat{U^cNE}, \\ b & \text{if } t \in \widehat{UNO}, \quad g \quad \text{if } t \in \widehat{UNE}. \end{cases} \tag{4}$$

As $\overline{N}$ is unique for any N, $x(S, \overline{N})$ and $y(S, \overline{N})$ are well-defined for any $S \in \Omega(N)$.

Let $\succsim$ be a social welfare order satisfying Pigou–Dalton Equity. We claim that the collection of sets $\Gamma \equiv \{N \in \Omega : y(N) \succ x(N)\}$ is non-Ramsey. We need to show that for each $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and $\Omega \setminus \Gamma$. For this, it is sufficient to show that for each $T \in \Omega$, there exists $S \in \Omega(T)$ such that either $T \in \Gamma$ or $S \in \Gamma$, with the either/or being exclusive. Let $T \equiv \{t_1, t_2, \ldots\}$. In the remaining proof we are concerned with infinite utility sequences $x(T, \overline{T}), y(T, \overline{T})$ and $x(S, \overline{T}), y(S, \overline{T})$ where $S \in \Omega(T)$. For ease of notation, we omit reference to $\overline{T}$. As the binary relation is complete, one of the following cases must arise: (a) $y(T) \succ x(T)$; (b) $x(T) \succ y(T)$; (c) $x(T) \sim y(T)$. Accordingly, we now separate our analysis into three cases.

(a) Let $y(T) \succ x(T)$; that is, $T \in \Gamma$. We drop t_1 from T to obtain $S = \{t_2, t_3, t_4, \ldots\}$. Hence $S \in \Omega(T)$. Let $T_1 \equiv \{2t_1 - 1, 2t_1 + 1, \ldots, 2t_2 - 3\}$ and $T_2 \equiv \{2t_1, 2t_1 + 2, \ldots, 2t_2 - 2\}$. Observe that
  - (A) for all $t \in T_1, x_t(T) = d > c = y_t(S)$;
  - (B) for all $t \in T_2, x_t(T) = e < f = y_t(S)$;
  - (C) for all the remaining $t \in \mathbb{N}, x_t(T) = y_t(S)$.
  - (D) for all $t \in \mathbb{N}, x_t(S) = y_t(T)$.

  Then for the generations $2t_1 - 1$ and $2t_1$,

  $$\begin{aligned} y_{2t_1-1}(S) &= c < d = x_{2t_1-1}(T) < x_{2t_1}(T) \\ &= e < f = y_{2t_1}(S), \quad \text{and} \quad c + f = d + e. \end{aligned}$$

  Similar inequalities hold for the pair of generations $\{2t_1 + 1, 2t_1 + 2\}, \ldots, \{2t_2 - 3, 2t_2 - 2\}$. Each of these pairs leads to PD improvements in $x(T)$ compared to $y(S)$. Since these are finitely many PD improvements, $x(T) \succ y(S)$ by PD. Also, $x(S) \sim y(T)$. Since $y(T) \succ x(T)$, we get

  $$x(S) \sim y(T) \succ x(T) \succ y(S).$$

  Thus, $x(S) \succ y(S)$ by transitivity of $\succsim$, and so $S \notin \Gamma$.

(b) Let $x(T) \succ y(T)$; that is, $T \notin \Gamma$. We drop t_1 and minimum number of t_{4n}, t_{4n+1} such that

  $$\begin{aligned} |\{2t_1 - 1, \ldots, 2t_2 - 2\}| \leq |\{2t_4 - 1, \ldots, 2t_5 - 2\} \cup \cdots \\ \cup \{2t_{4k} - 1, \ldots, 2t_{4k+1} - 2\}| \end{aligned}$$

  from T to obtain $S = \{t_2, t_3, t_6, t_7, t_{10}, t_{11}, \ldots\}$. Hence $S \in \Omega(T)$. Denote the set of coordinates $\{2t_1 - 1, 2t_1 + 1, \ldots, 2t_2 - 3\}$ by T_1, $\{2t_1, 2t_1 + 2, \ldots, 2t_2 - 2\}$ by T_2, $\{2t_4 - 1, 2t_4 + 1, \ldots, 2t_5 - 3, \ldots, 2t_{4k} - 1, \ldots, 2t_{4k+1} - 3\}$ by T_3 and $\{2t_4, 2t_4 + 2, \ldots, 2t_5 - 2, \ldots, 2t_{4k}, \ldots, 2t_{4k+1} - 2\}$ by T_4.
  - (i) For $x(T)$ and $y(S)$,
    - (A) for all $t \in T_1, x_t(T) = d > c = y_t(S)$;
    - (B) for all $t \in T_3, x_t(T) = a < b = y_t(S)$;
    - (C) for all $t \in T_2, x_t(T) = e < f = y_t(S)$;
    - (D) for all $t \in T_4, x_t(T) = h > g = y_t(S)$;
    - (E) for all the remaining coordinates, $x_t(T) = y_t(S)$.

    Following cases arise.
    - (I) For the generations $2t_1 - 1$ and $2t_4 - 1$,
      $$\begin{aligned} x_{2t_4-1}(T) &= a < b = y_{2t_4-1}(S) < y_{2t_1-1}(S) \\ &= c < d = x_{2t_1-1}(T), \quad \text{and} \\ &\quad a + d = b + c. \end{aligned}$$
      Similar inequalities hold for the pair of generations $\{2t_1 + 1, 2t_4 + 1\}, \ldots, \{2t_2 - 3, m\}$ where $m \in T_3$.
    - (II) For the generations $2t_1$ and $2t_4$,
      $$\begin{aligned} x_{2t_1}(T) &= e < f = y_{2t_1}(S) < y_{2t_4}(S) \\ &= g < h = x_{2t_4}(T), \quad \text{and} \quad e + h = f + g. \end{aligned}$$
      Similar inequalities hold for the pair of generations $\{2t_1 + 2, 2t_4 + 2\}, \ldots, \{2t_2 - 2, m + 1\}$ where $m + 1 = m' \in T_4$.
    - (III) For the generations $m' + 1, m' + 2$, and remaining generations[13] in $T_3 \cup T_4$,
      $$\begin{aligned} x_{m'+1}(T) &= a < b = y_{m'+1}(S) < y_{m'+2}(S) \\ &= g < h = x_{m'+2}(T), \quad \text{and} \\ &\quad a + h = b + g. \end{aligned}$$

    Each of these instances leads to PD improvements in $y(S)$ compared to $x(T)$ and there are finitely many of them. Hence, $y(S) \succ x(T)$ by PD.
  - (ii) For $x(S)$ and $y(T)$
    - (A) for all $t \in T_3, x_t(S) = a < b = y_t(T)$;
    - (B) for all $t \in T_4, x_t(S) = h > g = y_t(T)$;
    - (C) for all the remaining coordinates, $y_t(T) = x_t(S)$.

    The case of generations in T_3 and T_4 is similar to (b)(i)(III) above. Since these are finitely many PD improvements, $y(T) \succ x(S)$ by PD. Since $x(T) \succ y(T)$, we get $y(S) \succ x(T) \succ y(T) \succ x(S)$.

    Thus, $y(S) \succ x(S)$ by transitivity of $\succsim$, and so $S \in \Gamma$.

(c) Let $x(T) \sim y(T)$; that is, $T \notin \Gamma$. We drop t_1, t_2, t_3, and minimum number of t_{4n+2}, t_{4n+3} such that

  $$\begin{aligned} &|\{2t_1 - 1, \ldots, 2t_2 - 2\} \cup \{2t_3 - 1, \ldots, 2t_4 - 2\}| \\ &\quad \leq |\{2t_6 - 1, \ldots, 2t_7 - 2\} \cup \cdots \\ &\quad \cup \{2t_{4k+2} - 1, \ldots, 2t_{4k+3} - 2\}| \end{aligned}$$

  from T to obtain $S = \{t_4, t_5, t_8, t_9, \ldots\}$. Hence $S \in \Omega(T)$. Denote the set of coordinates $\{2t_2 - 1, 2t_2 + 1, \ldots, 2t_3 - 3\}$ by T_1, $\{2t_2, 2t_2 + 2, \ldots, 2t_3 - 2\}$ by T_2, $\{2t_1 - 1, 2t_1 + 1, \ldots, 2t_2 - 3\} \cup \{2t_3 - 1, 2t_3 + 1, \ldots, 2t_4 - 3\}$ by T_3, $\{2t_1, 2t_1 + 2, \ldots, 2t_2 - 2\} \cup \{2t_3, 2t_3 + 2, \ldots, 2t_4 - 2\}$ by T_4, $\{2t_6 - 1, 2t_6 + 1, \ldots, 2t_7 - 3, \ldots, 2t_{4k+2} - 1, \ldots, 2t_{4k+3} - 3\}$ by $\widehat{T}_1$, and $\{2t_6, \ldots, 2t_7 - 2, \ldots, 2t_{4k+2}, \ldots, 2t_{4k+3} - 2\}$ by $\widehat{T}_2$.
  - (i) For $x(S)$ and $y(T)$,

[12] If $n_1 = 1$, then $\{1, \ldots, 2(n_1 - 1)\} = \emptyset$. For illustration, for $N = \{1, 2, 3, 4, \ldots\}$, $\overline{N} = \{1, 2, 3, 4, 5, 6\}$ and the two utility streams are $x(N, \overline{N}) = \{d, e, c, f, d, e, a, h, b, g, \ldots\}$ and $y(N, \overline{N}) = \{c, f, d, e, c, f, b, g, a, h, \ldots\}$.

[13] The number of these generations is even.

(A) for all $t \in T_1, y_t(T) = d > c = x_t(S)$;
(B) for all $t \in \widehat{T}_2, y_t(T) = e < f = x_t(S)$;
(C) for all $t \in \widehat{T}_1, x_t(S) = a < b = y_t(T)$;
(D) for all $t \in \widehat{T}_2, x_t(S) = h > g = y_t(T)$;
(E) for all the remaining coordinates, $y_t(T) = x_t(S)$.

For $\widehat{T}_1, \widehat{T}_2$, PD improvements in $y(T)$ compared to $x(S)$ can be shown following the case (b)(i)(III) above. Also for T_1, T_2, PD improvements in $y(T)$ compared to $x(S)$ can be shown following the case (a) above. Since these are finitely instances of PD improvements, $x(S) \prec y(T)$ by PD.

(ii) For $x(T)$ and $y(S)$,
(A) for all $t \in T_3, x_t(T) = d > c = y_t(S)$;
(B) for all $t \in \widehat{T}_1, x_t(T) = a < b = y_t(S)$;
(C) for all $t \in T_4, x_t(T) = e < f = y_t(S)$;
(D) for all $t \in \widehat{T}_2, x_t(T) = h > g = y_t(S)$;
(E) for all the remaining coordinates, $y_t(S) = x_t(T)$.

Here, PD improvements in $y(S)$ compared to $x(T)$ can be shown following the case (b)(i) above. Since these are finitely many instances of PD improvements, $y(S) \succ x(T)$ by PD.

Since $x(T) \sim y(T)$, we get

$$y(S) \succ x(T) \sim y(T) \succ x(S).$$

Thus, $y(S) \succ x(S)$ by transitivity of $\succsim$, and so $S \in \Gamma$. □

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